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IDEAL SPACES OF MEASURABLE OPERATORS AFFILIATED TO A SEMIFINITE VON NEUMANN ALGEBRA

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Abstract: Suppose that $\mathscr{M}$ is a von Neumann algebra of operators on a Hilbert space $\mathscr{H}$ and τ is a faithful normal semifinite trace on $\mathscr{M}$. Let $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ be ideal spaces on $(\mathscr{M},\tau)$. We find when a τ-measurable operator X belongs to $\mathscr{E}$ in terms of the idempotent P of $\mathscr{M}$. The sets $\mathscr{E}+\mathscr{F}$ and $\mathscr{E}\cdot\mathscr{F}$ are also ideal spaces on $(\mathscr{M},\tau)$; moreover, $\mathscr{E}\cdot\mathscr{F}=\mathscr{F}\cdot\mathscr{E}$ and $(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}=\mathscr{E}\cdot\mathscr{G}+\mathscr{F}\cdot\mathscr{G}$. The structure of ideal spaces is modular. We establish some new properties of the $L_1(\mathscr{M},\tau)$ space of integrable operators affiliated to the algebra $\mathscr{M}$. The results are new even for the $*$-algebra $\mathscr{M}=\mathscr{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$ which is endowed with the canonical trace $\tau=\mathrm{tr}$.

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Introduction

The section of functional analysis, called noncommutative integration theory, is an important part of the theory of operator algebras. This article is devoted to noncommutative analogs of the classical methods for constructing function spaces. The beginning of the development of the corresponding aspect of noncommutative integration theory is related to the names of Segal and Dixmier, who in the early 1950s created a theory of integration with respect to a trace on a semifinite von Neumann algebra [1]. The results of these investigations found spectacular applications in the duality theory for unimodular groups and stimulated the progress of "noncommutative probability theory." The theory of algebras of measurable and locally measurable operators is rapidly developing and has interesting applications in various areas of functional analysis, mathematical physics, statistical mechanics, and quantum field theory.

In [2–4], Muratov introduced and investigated ideal spaces of measurable operators on a finite von Neumann algebra. They were also studied by Chilin in [5]. In the above-mentioned works, the ideal spaces serve primarily as the object of investigation. Recently, there have appeared publications in which they act as a tool. The foregoing demonstrates the relevance of (1), the search for new methods for constructing ideal spaces of measurable operators; (2), the development of a general theory of these spaces; and (3), the consideration of new particular examples.

Suppose that a von Neumann algebra $\mathscr{M}$ of operators acts on a Hilbert space $\mathscr{H}$, while τ is a faithful semifinite trace on $\mathscr{M}$. Let $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ be ideal spaces on $(\mathscr{M},\tau)$. Let us list the obtained results. Given a normal τ-measurable operator X and an idempotent $P\in\mathscr{M}$, we show that $X\in\mathscr{E}\Leftrightarrow XP+P^{\perp}X\in\mathscr{E}\Leftrightarrow PXP+P^{\perp}X\in\mathscr{E}\Leftrightarrow XP+P^{\perp}XP^{\perp}\in\mathscr{E}$ (Theorem 1). The condition of normality for X is substantial in Theorem 1 (Example 3). The sets $\mathscr{E}+\mathscr{F}$ and $\mathscr{E}\cdot\mathscr{F}$ are also ideal spaces on $(\mathscr{M},\tau)$; moreover, $\mathscr{E}\cdot\mathscr{F}=\mathscr{F}\cdot\mathscr{E}$ and $(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}=\mathscr{E}\cdot\mathscr{G}+\mathscr{F}\cdot\mathscr{G}$. The structure of ideal spaces is modular: if $\mathscr{E}\subset\mathscr{G}$ then $(\mathscr{E}+\mathscr{F})\cap\mathscr{G}=\mathscr{E}+(\mathscr{F}\cap\mathscr{G})$ (Theorems 2 and 3).

Let τ-measurable operators X, Y, and an idempotent $P\in\mathscr{M}$ be such that $XP-PY\in L_1(\mathscr{M},\tau)$. Then $\tau(XP-PY)=\tau(PXP-PYP)$ and for $X=Y$ we have $\tau([X,P])=0$ (Theorem 4). Let $\mathrm{Re}\,Y$

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